Defining a Double Team Semantics for Generalized Quantifiers (Extended Version)

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In this brief technical report we sketch a semantics for first-order logic with generalized quantifiers based on *double teams*. We also define the notion of a *generalized atom*. Such atoms can be used in order to define extensions of first-order logic with a team-based semantics. We then briefly discuss how our double team semantics relates to game semantics based approaches to extensions of first-order logic with generalized quantifiers.

1 Introduction

While the role of team semantics in IF logic [2] and dependence logic [3] is well known, team semantics should also be studied for its own sake.

Engström investigates generalized quantifiers in the context of team semantics in [1]. In that article, the scope of admissible generalized quantifiers is limited. In this short technical report we define a natural semantics based on *double teams* that works fine with any extension of first-order logic with a collection of generalized quantifiers.

We also consider extensions of first-order logic with *generalized atoms*. These are devices that can be used in order to declare properties that double teams must satisfy. While a sequence $x_1, ..., x_n$ of variables and a team give rise to an n-ary relation, a double team and two sequences of variables give rise to a pair of relations. With a generalized atom, one can assert that the pair of relations must satisfy some property. For example, an atom could assert that the first relation has some finite number k of tuples, while the other relation has $2k$ tuples.

We also briefly discuss the connection between team-based systems and game-theoretic approaches to semantics, and address some issues concerning possible applications of team semantics and logic in general.

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2 A Double Team Semantics

Let $\mathbb{N}_+$ denote the set of positive natural numbers, and let $\mathrm{VAR} = \{x_i \mid i \in \mathbb{N}_+\}$ be the set of exactly all first-order variable symbols. We shall occasionally use metavariables x, y, z instead of symbols $x_i \in \mathrm{VAR}$ in order to simplify notation.

Let $X \subseteq \mathrm{VAR}$ be a *finite*, possibly empty set. Let $\mathfrak{A}$ be a model with the domain A. We always assume that $A \neq \emptyset$. A function $f : X \longrightarrow A$ is called a variable assignment for the model $\mathfrak{A}$.

By $\overline{x}$ we denote a finite nonempty tuple of variable symbols, i.e., a tuple in VAR^n for some $n \in \mathbb{N}_+$. We also use $\overline{y}, \overline{z}, \overline{x}_1, \overline{x}_2$, etc., to refer to finite nonempty tuples of variable symbols.

Let $\overline{a}$ be a finite nonempty tuple. We let $\overline{a}(k)$ denote the k-th member of the tuple. When we write $u \in \overline{a}$, we mean that u is a member of the tuple $\overline{a}$, i.e., if $\overline{a} = (a_1, ..., a_n)$, then $u \in \overline{a}$ iff $u \in \{a_1, ..., a_n\}$. If f is a function mapping into some set S^k of tuples of the length $k \in \mathbb{N}_+$, then f_i denotes a function with the same domain as f and defined such that

$$f_i(x) \;=\; \big(f(x)\big)(i),$$

i.e., f_i is the i-th coordinate function of f, and therefore $f_i(x)$ is the i-th coordinate value of $f(x)$.

Let s be a variable assignment with the domain X and for the model $\mathfrak{A}$. Let $\overline{x} \in \mathrm{VAR}^n$ be a tuple of variables and $\overline{a} \in A^n$ a tuple of the same length as $\overline{x}$. We let $s[\overline{x}/\overline{a}]$ denote the variable assignment for $\mathfrak{A}$ with the domain $X \cup \{\, x \mid x \in \overline{x} \,\}$ defined as follows.

1. $s[\overline{x}/\overline{a}](y) \;=\; \overline{a}(k) \quad \text{if} \quad y = \overline{x}(k),$
2. $s[\overline{x}/\overline{a}](y) \;=\; s(y) \quad \text{if} \quad y \notin \overline{x}.$

Let $T \subseteq \mathcal{P}(A^n)$, where $\mathcal{P}$ denotes the power set operator. We define

$$s[\,\overline{x}/T\,] \;=\; \{\; s[\,\overline{x}/\overline{a}\,] \mid \overline{a} \in T \;\}.$$

Note that $s[\,\overline{x}, \emptyset\,] = \emptyset$.

Let $X \subseteq \mathrm{VAR}$ be a finite, possibly empty set of first-order variable symbols. Let U be a set of variable assignments $f : X \longrightarrow A$. Such a set U is a *team* with the domain X for the model $\mathfrak{A}$. Note that the empty set is a team for $\mathfrak{A}$, as is the set $\{\emptyset\}$ containing only the empty variable assignment. The domain X associated with the empty team is not unique.

Let V be a nonempty team with the domain X for the model $\mathfrak{A}$. Let $n \in \mathbb{N}_+$, and let $f : V \longrightarrow \mathcal{P}(A^n)$ be a function. We write

$$V[\,\overline{x}/f\,] = \bigcup_{s \,\in\, V} s[\,\overline{x}/f(s)\,].$$

If $V = \emptyset$, i.e., V is the empty team, then $V[\overline{x}/f] = \emptyset$.

Let $n \in \mathbb{N}_+$. If $f : V \longrightarrow \mathcal{P}(A^n)$ is a function, then $f' : V \longrightarrow \mathcal{P}(A^n)$ denotes the function defined such $f'(s) = A^n \setminus f(s)$ for all $s \in V$. Below when applying the notation f', the set A^n and the function f are always known from the context, so there is no ambiguity related to the definition of f'.

Let $(i_1, ..., i_n)$ be a non-empty sequence of positive integers. A generalized quantifier of the type $(i_1, ..., i_n)$ is a class $\mathcal{C}$ of structures $(A, B_1, ..., B_n)$ such that the following conditions hold.

1. $A \neq \emptyset$.
2. For each $j \in \{1, ..., n\}$, we have $B_j \subseteq A^{i_j}$.
3. If $(A', B_1', ..., B_n') \in \mathcal{C}$ and if there is an isomorphism $f : A' \longrightarrow A''$ from $(A', B_1', ..., B_n')$ to another structure $(A'', B_1'', ..., B_n'')$, then we have $(A'', B_1'', ..., B_n'') \in \mathcal{C}$.

Let Q be a generalized quantifier of the type $(i_1, ..., i_n)$. We let $\overline{Q}$ denote the generalized quantifier of the type $(i_1, ..., i_n)$ defined such that

$$\overline{Q} \;=\; \{\ (A, C_1, ..., C_n) \mid (A, C_1, ..., C_n) \notin Q\ \}.$$

Let $\mathfrak{A}$ be a model with the domain A. We define $Q^{\mathfrak{A}}$ to be the class

$$\{\ (B_1, ..., B_n) \mid (A, B_1, ..., B_n) \in Q\ \}.$$

Similarly, we define

$$\overline{Q}^{\mathfrak{A}} \;=\; \{\ (B_1, ..., B_n) \mid (A, B_1, ..., B_n) \in \overline{Q}\ \}.$$

Below we shall define a semantics based on *ordered sets of teams*. Going this step higher from teams to ordered sets of teams enables a flexible way of encoding information that will be used in order to define a natural team-based semantics for first-order logic extended with generalized quantifiers. In fact, in the investigations below, we shall concentrate on *double teams*, i.e., ordered pairs of teams. In ordinary team semantics for ordinary first-order logic, the *background intuition* is that a team satisfies a formula iff every member of the team satisfies it. In the double team semantics, the *background intuition* is that a double team (U, V) satisfies a first-order formula iff every assignment in the team U satisfies it, and furthermore, every assignment in the team V *falsifies* the formula. Therefore, the set set of witnesses for a quantified formula $Qx\,\varphi(x)$ has to be the set of *exactly all* values a such that $\varphi(a)$ holds. A subset will not do. This is the main intuition behind the systems of team semantics and game semantics below.

The definition for atoms and connectives is as follows.

$$
\begin{array}{lcl}
\mathfrak{A}, (U, V) \models t_1 = t_2 & \Leftrightarrow & \forall s \in U\big(\mathfrak{A}, s \models_{\mathrm{FO}} t_1 = t_2\big) \text{ and} \\
 & & \forall s \in V\big(\mathfrak{A}, s \not\models_{\mathrm{FO}} t_1 = t_2\big). \\
\mathfrak{A}, (U, V) \models R(t_1, ..., t_m) & \Leftrightarrow & \forall s \in U\big(\mathfrak{A}, s \models_{\mathrm{FO}} R(t_1, ..., t_m)\big) \text{ and} \\
 & & \forall s \in V\big(\mathfrak{A}, s \not\models_{\mathrm{FO}} R(t_1, ..., t_m)\big). \\
\mathfrak{A}, (U, V) \models \neg\varphi & \Leftrightarrow & \mathfrak{A}, (V, U) \models \varphi. \\
\mathfrak{A}, (U, V) \models (\varphi \vee \psi) & \Leftrightarrow & \mathfrak{A}, (U_1, V) \models \varphi \text{ and } \mathfrak{A}, (U_2, V) \models \psi \\
 & & \text{for some teams } U_1 \text{ and } U_2 \text{ such that} \\
 & & U = U_1 \cup U_2.
\end{array}
$$

For a generalized quantifier Q of the type $(i_1, ..., i_n)$, we define

$$\mathfrak{A}, (U, V) \models Q\overline{x}_1, ..., \overline{x}_n(\varphi_1, ..., \varphi_n)$$

if and only if there exists a function $f : U \longrightarrow Q^{\mathfrak{A}}$ such that

$$
\begin{gathered}
\mathfrak{A}, \big(\, U[\,\overline{x}_1/f_1],\, U[\,\overline{x}_1/{f_1}'\,]\,\big) \models \varphi_1, \\
\vdots \\
\mathfrak{A}, \big(\, U[\,\overline{x}_n/f_n],\, U[\,\overline{x}_n/{f_n}'\,]\,\big) \models \varphi_n,
\end{gathered}
$$

and there exists a function $g : V \longrightarrow \overline{Q}^{\mathfrak{A}}$ such that

$$\mathfrak{A}, \big(V[\overline{x}_1/g_1],\, V[\overline{x}_1/g_1{}']\big) \models \varphi_1,$$
$$\vdots$$
$$\mathfrak{A}, \big(V[\overline{x}_n/g_n],\, V[\overline{x}_n/g_n{}']\big) \models \varphi_n.$$

Proposition 2.1. *Let φ be a formula of first-order logic, possibly extended with generalized quantifiers. Let (U, V) be a double team. Then*

$$\mathfrak{A}, (U, V) \models \varphi \quad \textit{iff} \quad \forall s \in U \forall t \in V \big(\mathfrak{A}, s \models_{\mathrm{FO}} \varphi \text{ and } \mathfrak{A}, t \not\models_{\mathrm{FO}} \varphi\big).$$

Proof. The claim of the proposition is established by a straightforward induction on the structure of formulae. □

3 Generalized Atoms and Operators

Let V be a team with the domain X for the model $\mathfrak{A}$, and let $k \in \mathbb{N}_+$. Let $t_1, ..., t_k$ be terms whose constant and function symbols are part of the vocabulary of $\mathfrak{A}$, and assume that X contains all variable symbols that occur in the terms. Define

$$\mathrm{Rel}\big(\mathfrak{A}, V, (t_1, ..., t_k)\big) \;=\; \{\; \big(s(t_1), ..., s(t_k)\big) \mid s \in V \;\},$$

where s extends to interpret terms with constant and function symbols in the obvious way. If V is empty, then the obtained relation is the empty relation.

Let $n, m \in \mathbb{N}$ such that $n + m > 0$. Let Q be a generalized quantifier of the type $(i_1, ..., i_{n+m})$. Extend the syntax of first-order logic with atomic expressions of the type

$$A_{Q,n}(\bar{t}_1, ..., \bar{t}_n \,;\, \bar{t}_{n+1}, ..., \bar{t}_{n+m}),$$

where each $\bar{t}_j$ is a tuple of terms of the length i_j.

Extend the double team semantics such that

$$\mathfrak{A}, (U, V) \models A_{Q,n}(\bar{t}_1, ..., \bar{t}_n \,;\, \bar{t}_{n+1}, ..., \bar{t}_{n+m})$$

if and only if

$$\big(\mathrm{Rel}(U, \mathfrak{A}, \bar{t}_1), ..., \mathrm{Rel}(U, \mathfrak{A}, \bar{t}_n), \mathrm{Rel}(V, \mathfrak{A}, \bar{t}_{n+1}), ..., \mathrm{Rel}(V, \mathfrak{A}, \bar{t}_{n+m})\big) \in Q^{\mathfrak{A}}.$$

The generalized quantifier Q and the number n define a *generalized atom* $A_{Q,n}$ of the type

$$\big((i_1, ..., i_n), (i_{n+1}, ..., i_{n+m})\big).$$

Note that types of generalized quantifiers are tuples and types of generalized atoms are pairs of tuples.

Generalized atoms with one part for positive information and another for negative information can be used in order to define double team based logics with *natural duality properties*. In addition to generalized atoms, one can also consider *generalized connectives* in the team based higher order setting. We shall give a functional definition of a framework where one can easily and relatively canonically define various kinds of natural higher order operators.

In the rest of this section, a model is a tuple $(S, P_1, ..., P_n)$, where S is a set and P_i are predicates interpreted as *subsets* of S, i.e., $P_i \subseteq S$. Also an infinite set of predicates could be considered. Define the language

$$\varphi \ ::= \ P_i \mid (\varphi \vee \psi) \mid \neg\, \varphi \mid (\varphi \overline{\vee} \psi) \mid \overline{\neg}\, \varphi,$$

where P_i are predicates. (We shall not differentiate between predicates and predicate symbols here.)

Define the following.

$$\begin{aligned}
||P_i|| &= \{ \ V \subseteq S \mid V \subseteq P_i \ \},\\
||(\varphi \vee \psi)|| &= \{ \ V \subseteq S \mid V = V_1 \cup V_2 \text{ for some } V_1 \in ||\varphi|| \text{ and } V_2 \in ||\psi|| \ \},\\
||\neg\, \varphi|| &= \{ \ V \subseteq S \mid \text{for all } U \in ||\varphi||,\ V \subseteq (S \setminus U) \ \},\\
||(\varphi \overline{\vee} \psi)|| &= ||\varphi|| \cup ||\psi||,\\
||\overline{\neg}\, \varphi|| &= \mathcal{P}(S) \setminus ||\varphi||.
\end{aligned}$$

In this setting we are considering logic over a collection S of *worlds* $w \in S$ (or *states* or whatever objects $w \in S$). We have *local connectives* $\vee$ and $\neg$ and *global connectives* $\overline{\vee}$ and $\overline{\neg}$. We define that φ is valid in $U \subseteq S$ iff $U \in ||\varphi||$. For example commutativity is not valid in the class of groups but is valid in the class of Abelian groups. We can also define, for example, that that φ is true in $w \in S$ iff $\{w\} \in ||\varphi||$.

We can extend this logic by generalized operators. A generalized operator f of the arity $n \in \mathbb{N}$ is a mapping such that for each model with domain S and sets $U_1, ..., U_n \subseteq \mathcal{P}(S)$, the map returns a set $U \subseteq \mathcal{P}(S)$. Furthermore, f is closed under isomorphisms: Let $(S, U_1, ..., U_n)$ and $(T, V_1, ..., V_n)$ be tuples such that conditions $U_1, ..., U_n \subseteq \mathcal{P}(S)$ and $V_1, ..., V_n \subseteq \mathcal{P}(T)$ hold. Here S and T are arbitrary sets. If some bijection $g : S \to T$ satisfies $g(U_i) = V_i$ for all $i \in \{1, ..., n\}$, then we have $g(f(S, U_1, ...U_n)) = f(T, V_1, ..., V_n)$. Obviously, semantics for the formula $f(\varphi_1, ..., \varphi_n)$ is defined such that

$$||f(\varphi_1, ..., \varphi_n)|| \ = \ f(||\varphi_1||, ..., ||\varphi_n||).$$

The logic defined above resembles team logic [3] and modal dependence logic [4]. Both of those systems have problems with the nesting of operators. There are no such problems in the above framework. Also, the framework works rather smoothly without the use of normal forms. This is holds for double team semantics too.

It would be natural to extend the above logic by higher order predicates $\overline{P}_i$ denoting subsets of $\mathcal{P}(S)$ rather than subsets of S. One could also consider connectives for further levels of the power set hierarchy. Also, it would be an interesting excercise to investigate what kind of a degree of canonicity the above semantics has.

The functional semantics given above can of course be done in the setting of first-order logic as well. Essentially this involves operators that output a set of assignments on the input of a tuple of sets of assignments. This is because in the setting of standard first-order logic, the extension of a first-order formula (in a fixed model) is the set of assignments satisfying the formula. This all leads naturally to Scott-Montague semantics, general compositional semantics, and issues concerning cylindric set algebras. Of course also operators on sets of teams can be considered, if desired.

4 Team vs. Game Semantics

Let $\mathfrak{A}$ be a model with the domain A. Consider first-order logic with a collection $\mathcal{Q}$ of generalized quantifiers of the type (1). Let f be an assignment function mapping a finite set of first-order variable symbols into A. We define a semantic game $(\mathfrak{A}, \varphi, f, *)$, where $* \in \{I, II\}$, between two players I and II. Here we assume that the assignment f interprets all the free variables in φ.

If a position $(\mathfrak{A}, \psi, g, *)$ is reached, where ψ is an atomic formula, then $*$ wins the game if $\mathfrak{A}, g \models_{\mathrm{FO}} \psi$. If $\mathfrak{A}, g \not\models_{\mathrm{FO}} \psi$, then player $\overline{*} \in \{I, II\} \setminus \{*\}$ wins the game.

Assume we have reached the position $(\mathfrak{A}, \neg\psi, g, *)$ in the game. The game continues from the position $(\mathfrak{A}, \psi, g, \overline{*})$. If a position $(\mathfrak{A}, \psi \vee \psi', g, *)$ is reached, The player $*$ chooses a formula $\chi \in \{\psi, \psi'\}$, and the game continues from the position $(\mathfrak{A}, \chi, g, *)$.

Assume we have reached the position $(\mathfrak{A}, Qx\psi, g, *)$ in the game. The game continues as follows.

1. The player $*$ chooses a set $S \in Q^{\mathfrak{A}}$. The intuition is that the player $*$ claims that S is the set of *exactly all* values that witness ψ.

2. The player $\overline{*}$ chooses either the set S or its complement $A \setminus S$.

   (a) If $\overline{*}$ chooses S, then $\overline{*}$ also chooses an element $s \in S$, and the game continues from the position $(\mathfrak{A}, \psi, g[x/s], *)$. In this case the intuition is that the player $\overline{*}$ is opposing the claim that s verifies ψ. (If $S = \emptyset$ and the player $\overline{*}$ chooses S, the player $\overline{*}$ looses the game.)

   (b) If $\overline{*}$ chooses $A \setminus S$, then $\overline{*}$ also chooses an element $s \in A \setminus S$. The game continues from the position $(\mathfrak{A}, \psi, g[x/s], \overline{*})$. In this case the player $\overline{*}$ becomes the verifier in the game. The intuition is that the player $\overline{*}$ is opposing the claim that *s falsifies* ψ. (If $A \setminus S = \emptyset$ and the player $\overline{*}$ chooses $A \setminus S$, the player $\overline{*}$ looses.)

This is a conservative semantics for first-order logic with generalized quantifiers (of the type (1) in this case, for the sake of simplicity). In some sense, game semantics can be regarded as a fundamental system of semantics linking a language to its action-based acquisition process. Team semantics is a mathematical tool for keeping track of game histories. It would be an interesting exercise to attempt to identify a general and canonical translation of (not necessarily logic-related) game-based systems to team-based ones, i.e., ones that keep track of possible game histories. The game-based and team-based semantics defined above, or their close variants, could probably be canonically linked in this way.

It is easy to define natural variants of the above systems of semantics. In the game semantics above, quantifiers are treated in a maximal fashion: the verifying player must choose exactly all members in the domain that witness the quantified formula. One could do the analogous thing with connectives:

at a position for $\psi \vee \psi'$, the verifying player would have to choose exactly the subset of $\{\psi, \psi'\}$ of formulae that are true, and that subset would have to be nonempty. The opposing player would then choose a formula either from the set chosen by the verifier or its complement set. A choice from the complement set would result in a switch of roles. Also, the disjunction clause of the double team semantics can easily be seen to allow for alternative definitions.

5 Concluding Remarks

Team semantics is a second-order semantics for first-order languages. It is second-order in the sense that instead of considering single assignments, as in the traditional semantics, it considers sets of assignments. However, team semantics is conservative with respect to traditional Tarskian semantics on ordinary first-order formulae. Indeed, it is a good question what can be achieved by introducing a system of semantics that generalizes some traditional system in some conservative way. This is one of the interesting points of team semantics. It is a natural way of generalizing classical first-order semantics, and it naturally links the game-theoretical semantics of first-order logic with Tarskian semantics.

Above we have studied a double team semantics for first-order logic with generalized quantifiers. It is an intriguing question what could be achieved by considering a semantics based on, say, ordered sets of teams, or sets of ordered sets of teams, etc. While such questions can appear undermotivated, they can be regarded as basic research in logic not motivated by applications; basic research of logic hardly seems unimportant when reflected against the development of modern science in general. Even though not directly related to team semantics, let us digress to briefly discuss possible uses of logic and different systems of semantics in future scientific contexts.

Today, physics is an impressive, huge body of statements, described mostly in an informal mathematical language. Theories are informal. Axiomatized ones are mainly informally axiomatized, i.e., it is often not quite clear what the background formal language would be, and what the signature —if any— in that language would be. Furthermore, *having* some exact axiomatization is one thing, but one can also ask how *relevant* a particular axiomatization is. Let us consider a situation where we have a sensible, decidable theory[1] T of some system of science. From the point of view of being able to access assertions of theories, this is better than having an undecidable theory. So, T is an accessible theory. However, assume that the complexity of T is very high. Assume we have another theory, $T' \subseteq T$, describing the same phenomenon as T, and assume that T' is of a lower complexity.[2] Assume that both T and T' are non-trivial theories describing our system. Now, neither one of the theories is in any obvious sense better than the other one. There remains a *trade-off* between information content and the speed of use. And, in addition to speed issues, there are rather *fundamental qualitative differences* between decidable theories, undecidable but recursively enumerable theories, and theories that are

[1]By a theory we mean simply a set of formulae; no closure under consequence is assumed here.

[2]This is always possible, consider the theory $\{\forall x(x = x)\}$, for example.

not even recursively enumerable.[3] The traditional approach to science still does not worry much about such issues. Ultimately these kinds of questions will be faced. Then the naive approach, where a single theory is being built, will not suffice. Then formal logic and informatics can help.

When decidability is not required, ZFC is, of course, one currently paradigmatic axiom system within which various different systems can be easily interpreted. The theoremhood of ZFC is recursively enumerable, and the system has a kind of a verification property: if φ is a theorem of ZFC, then it has a finite, formal proof using only ZFC axioms together with primitives of some first-order proof system. Once we know a formal proof of a theorem φ of ZFC, then, to question φ, one must question ZFC axioms or primitives of our first-order proof system. Therefore, φ is, in a sense, relatively strongly verified. Therefore, by staying in ZFC, one eliminates a lot of philosophical worry. The formal system is, on the formal level, or in itself, discrete, clear and finitary in a relatively strong sense.

However, the theoremhood of ZFC is not decidable, and there is also the independence phenomenon associated with this system. Let us finish our little digression by briefly discussing the independence issue.

So, there exist formulae φ such that ZFC $\nvdash \varphi$ and ZFC $\nvdash \neg\varphi$, assuming consistency. There are formulae whose truth the formal system does not decide. However, it is another matter how independence of some formula φ should be interpreted. For example, it is straightforward to construct a computable reduction F from a set of formal specifications of Turing machines to first-order logic such that machine M halts on the empty input iff ZFC $\vdash F(M)$.[4] There is then, of course, some machine M' such that ZFC $\nvdash F(M')$ and ZFC $\nvdash \neg F(M')$. The machine M' does not halt on the empty input. One can ask whether halting of M' is undecided by ZFC on some fundamental level. It is true that ZFC $\nvdash F(M')$ and ZFC $\nvdash \neg F(M')$, but one can define a new, computable reduction F' such that for each machine M, M halts on the empty input iff ZFC $\vdash F'(M)$, and furthermore, ZFC $\vdash \neg F'(M')$. This is always possible by simply defining $F'(M) := F(M)$ for all M except for M', and $F'(M') := \neg\forall x(x = x)$. Of course one may then claim that our new reduction F' is not *canonical* or *natural* in some sense. On the other hand, it is also *conceivable* that we have, simply by observing the specification of M', directly concluded that this machine does not halt on the empty input, and therefore decided to define a map F' that maps M' directly to $\neg\forall x(x = x)$. While independence of the formal statement $F(M')$ of the axioms of ZFC is unquestionable, we may ask what the meta-level implication of that formal fact really is. One can always wonder what the *canonical formalization* of an unformalized assertion should be, and why. In the absence of the Axiom of Choice, Dedekind infinity and infinity based on natural numbers provide a striking example of a pair of non-equivalent formalizations of the same meta-level concept.

Independence of a statement and its negation implies that in some sense the system described is not fully specified. This *can* be regarded as a flaw. However, often a perfectly sensible approach can be based on the background assumption that the theoremhood of an object of reflection *should not* be determinate in the

[3]It should also be noted that currently standard complexity measures do not always necessarily directly reflect complexity phenomena faced in real-life situations.

[4]It is very easy to construct a natural map F when one allows for more predicate symbols than $\in$.

first place. In a possible context of informal mathematics, we could suddenly be faced with the question whether $\forall x(x \notin x)$ holds.[5] Then, an anti-realist could claim that since the notion of a set is not intersubjectively determined well enough, we do not have $\forall x(x \notin x)$. Could we then conclude that therefore, since we do not have $\forall x(x \notin x)$, we have $\neg\forall x(x \notin x)$? One should now ask what the precise meanings of the related claims are. The logic introduced in Section 3 provides a framework for related investigations, with the set of worlds being interpreted as the class C of all $\{\in\}$-models. Some proper subclass T of C is interpreted as the set of all acceptable models of set theory. Let $\forall(x \notin x)$ be a predicate symbol in the related framework, interpreted to be true in those worlds of C where x connects to itself via the relation $\in$. If we accept that $\forall x(x \notin x)$ is not valid in T, we can easily accept that $\overline{\neg}\forall x(x \notin x)$ is valid in T. Continuing the use of the connectives of Section 3, we do *not* necessarily have to accept that $\neg\forall x(x \notin x)$ is valid in T. Similarly, if we accept that neither $\forall x(x \notin x)$ nor $\neg\forall x(x \notin x)$ is valid in T, we do *not* have to accept that $\forall x(x \notin x) \vee \neg\forall x(x \notin x)$ is *not* valid in T, but we can quite easily accept that $\forall x(x \notin x)\overline{\vee}\neg\forall x(x \notin x)$ is *not* valid in T, and that therefore $\overline{\neg}\big(\forall x(x \notin x)\overline{\vee}\neg\forall x(x \notin x)\big)$ is valid in T. Also, $\forall x(x \notin x)\overline{\vee}\,\overline{\neg}\forall x(x \notin x)$ is valid in T, and so is $\forall x(x \notin x) \vee \overline{\neg}\forall x(x \notin x)$.[6] It is indeed a tempting idea to keep connectives Boolean and use modal operators to talk about modes of validity (and satisfiability).

Questions concerning the excluded middle are interesting indeed, and problems with mathematical realism can appear quite real. The notion of *definition* is central in this context. Everybody is familiar with situations where some novel, intuitively relatively clear notion is used, but still, the notion is not completely fixed and opinions about its precise canonical properties can differ from person to person. Such situations exist in mathematical practise, and one may wonder whether less novel mathematical notions should be immune to vagueness in the twilight zone? Is there a real qualitative difference here? Are human cognitive mechanisms so uniform, that there always exists exactly one canonical outcome of a sufficiently clearly initiated notion construction process?[7] Realist perspectives can feel dubious, but obviously anti-realist perspectives have their problems as well. And so do hybrid perspectives. For example, sometimes one hears about realism concerning arithmetic and simultaneous anti-realism concerning sets. This kind of a hybrid perspective appears problematic for example when higher order languages are used in order to talk about arithmetic. The talk can really be more about sets there.

[5] This is of course a theorem of ZFC.

[6] Without care one could be fooled into concluding from the two premises asserting that neither $\forall x(x \notin x)$ nor $\neg\forall x(x \notin x)$ is valid, that therefore—by the first premise— the formula $\neg\forall x(x \notin x)$ is valid, and by the second premise $\forall x(x \notin x)$ is valid, and hence $\neg\forall x(x \notin x) \wedge \forall x(x \notin x)$ is valid.

[7] Most mathematicians would consider the role of the Axiom of Union in set theory rather different from the role of the associative law in group theory. The first one is an axiom and the second one part of a definition. Is the difference between definitions and axioms fundamental or only a matter of degree? Something as fundamental as the universal quantifier could at least conceivably be misunderstood to stand for some other quantifier that is equivalent to $\forall$ in restriction to models of size less that 2^{1000}. Perhaps somebody with rather unique and unnecessarily complex cognitive faculties could conclude from his first sensible concrete experiences that this is what "all" means. Or perhaps not really! But absolutely everything can be plausibly argued for. And absolutely everything can be plausibly argued againts. There is no limit to that! Too bad for those seeking rock-solid intellectual clarity, control and stability.

Usually in mathematics the implicit background assumption seems to be that there is exactly one model within the framework of which the investigations are carried out. Even when one is not ready to accept that all related notions are completely specified (such as the notion of set), one usually still considers a single, fixed (but not fully understood) background model. For example, when defining the semantics of second-order logic, one usually asserts that $M \models \neg\varphi$ iff not $M \models \varphi$. This definition would lead to rather surprising issues if "not" was to mean contradictory negation, as usual, and, for each formula χ, "$M \models \chi$" was to suddenly mean that χ is true no matter what interpretation of the notion of set—out of some collection of acceptable background models— is meant. For example, any formula that is supposed to assert the Continuum Hypothesis can be considered in order to understand the questions that could arise here. Above we have developed an approach to connectives that can elucidate logical phenomena involving assertions about truth in a single model as well as validity in a collection of models.

Different formal systems have different uses, strengths and weaknesses. A theory T can have a higher information content than a related theory T', but T' may be of a lower complexity. Different approaches to semantics lead to different kinds of logic-based investigations of formal systems. Viewing phenomena from a variety of perspectives, rather than a single one, can be more fruitful and delightful than insisting on a single orthodox approach. Team semantics is an interesting extension of standard Tarskian semantics, and it also offers a nice complementary perspective on the game-theoretic semantics of first-order logic. In this article we have introduced a canonical double team semantics for first-order logics with generalized quantifiers, and also defined the notion of a generalized atom. It shall be interesting to see whether these considerations offer a fruitful platform for future developments of systems based on team semantics.

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